Assignment 10.

This homework is due *Thursday*, Nov 12.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 11.

1. General Lebesgue integral. Quick reminder

For an arbitrary measurable function $f : E \to \mathbb{R} \cup \pm\infty$, define its Lebesgue integral over E by

$$\int_E f = \int_E f^+ - \int_E f^-, \text{ provided at least one of values } \int_E f^+, \int_E f^- \text{ is finite.}$$

In the case when $\int_E f$ is finite (i.e. both $\int_E f^+, \int_E f^-$ are finite) the function f is said to be Lebesgue integrable over E.

The integral defined above is linear, monotone and domain additive. Key statements about Lebesgue integral are:

Fatou's Lemma. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then $\int_E f \le \liminf \int_E f_n$.

Monotone Convergence Theorem. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then $\int_E f = \lim \int_E f_n$.

The Lebesgue Dominated Convergence Theorem. Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g integrable over E s.t. $|f_n| \le g$ on E for all n. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim \int_E f_n = \int_E f$.

2. Exercises

(1) (4.4.29+) For a locally bounded (therefore bounded on closed bounded sets by Heine–Borel) measurable function f on $[1, \infty)$, define $a_n = \int_n^{n+1} f$ for each $n \in \mathbb{N}$.
   (a) Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges?
   (b) Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely? (*Hint:* Still no.)
   (c) Is the assertion in the previous item true if we additionally require f to be nonnegative on $[1, \infty)$? (*Hint:* Use the Monotone Convergence Theorem.)

(2) (a) (4.4.34) Let f be a *nonnegative* measurable function on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{[-n,n]} f = \int_{\mathbb{R}} f.$$

   (*Hint:* Use Monotone Convergence theorem.)
   Explain why one cannot use Lebesgue Dominated Convergence theorem here.
   (b) Prove that the same equality holds if f is arbitrary *integrable* over $\mathbb{R}$ function. (*Hint:* Use the Dominated Convergence.)

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(3) (4.5.37) Let f be integrable function on E. Show that for each $\varepsilon > 0$, there is a natural number N for which if $n \geq N$, then $\left|\int_{E_n} f\right| < \varepsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$. (*Hint:* Use continuity of integration; or countable domain additivity of integration.)

(4) (4.5.38i) Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = (-1)^n/n$ for $n \leq x < n+1$, $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} \int_1^n f$ exists while f is not integrable over $[1, \infty)$. Does this contradict continuity of integration?
COMMENT. By the way, this highlights difference between improper Riemann integral and Lebesgue integral.

3. Metric spaces. Quick reminder

Metric space is a pair (X, ρ), where X is a nonempty set and ρ is a function $\rho : X \times X \to \mathbb{R}$, called metric, such that $\forall x, y, z \in X$

(1) $\rho(x, y) \geq 0$,
(2) $\rho(x, y) = 0$ if and only if $x = y$,
(3) $\rho(x, y) = \rho(y, x)$,
(4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

A function ρ that satisfies (1),(3),(4), but not necessarily the "only if" part of (2), is called a *pseudometric.*

Normed linear space is a pair $(V, \|\cdot\|)$, where V is a linear space and $\|\cdot\|$ is a function $\|\cdot\| : V \to \mathbb{R}$, called norm, such that $\forall u, v \in V$ and $\forall \alpha \in \mathbb{R}$,

(1) $\|u\| \geq 0$,
(2) $\|u\| = 0$ if and only if $u = 0$,
(3) $\|u + v\| \leq \|u\| + \|v\|$,
(4) $\|\alpha u\| = |\alpha|\|u\|$.

A function $\|\cdot\|$ that satisfies (1),(3),(4), but not necessarily the "only if" part of (2), is called a *pseudonorm.*

Every norm induces a metric via $\rho(u, v) = \|u - v\|$.

4. More exercises

(5) (9.1.4+)
  (a) Let $X = C[a, b]$. Show that $\|f\|_1 = \int_{[a,b]} |f|$ is a norm.
  (b) Show that the norm above is not equivalent to $\|f\|_\infty$. That is, show that there are no constants $c_1, c_2 > 0$ such that $\forall f \in C[a, b]$, $c_1\|f\|_1 \leq \|f\|_\infty \leq c_2\|f\|_1$. Reminder: $\|f\|_\infty = \max\limits_{x\in[a,b]} \{|f(x)|\}$.

(6) ($\sim$9.1.5) Reminder: for sets A, B, their *symmetric difference* is defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
The Nikodym Metric. Let E be a Lebesgue measurable set of real numbers of finite measure. Let X be the set of Lebesgue measurable subsets of E, and m Lebesgue measure. For $A, B \in X$ define $\rho(A, B) = m(A \triangle B)$. Show that ρ is pseudometric, but not a metric, on X. Show that $\rho(A, B) = \int_E |\chi_A - \chi_B|$.

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(7) (a) (9.1.6) Show that for $a, b, c \geq 0$, if $a \leq b + c$, then $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$. (*Hint:* Straightforward way: multiply by common denominator; sneaky way: use concavity/convexity of $x/(1+x)$.)

(b) Let (X, ρ) be an arbitrary metric space. Prove that $(X, \frac{\rho}{1+\rho})$ is also a metric space.

NOTE. This turns any metric space into a *bounded* metric space.

(c) (9.1.10) Let $\{(X_n, \rho_n)\}$ be a countable collection of metric spaces. Show that ρ_* defines a metric space on the Cartesian product $\prod_{n=1}^{\infty} X_n$, where for points $x = \{x_n\}, y = \{y_n\} \in \prod_{n=1}^{\infty} X_n$,

$$\rho_*(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

5. Extra Problem

(8) Show that *pointwise* convergence in $C[0, 1]$ is not metrizable. That is, show that there does not exist a metric ρ on $C[0, 1]$ such that for $f_n, f \in C[0, 1]$, a sequence $\{f_n\}$ converges pointwise to f if and only if $\lim_{n\to\infty} \rho(f_n, f) = 0$.